Mathematical Concepts Revision Week3

Basic Algebra

Simplifying expressions

e.g. $5a + 2b + 2a - 3b$

$= 7a - b$

e.g. $3b \times 4b$

$= 12b^2$

e.g. Find x → $3x = -12$

$x = -4$

(product rule)

Rule

$$a^x \times a^y = a^{x+y}$$

e.g. $a^3 \times a^5 = a^{3+5}$

$= a^8$

Rule (Quotient Rule)

$$\frac{a^x}{a^y} = a^{x-y}$$

e.g.

$$\frac{a^5}{a^4} = a$$

Rule (power rule)

$$(a^x)^y = a^{xy}$$

e.g. $(a^3)^4 =$

a^{12}

Rule (power of a product rule)

$$(ab)^x = a^x b^x$$

e.g $(xy)^4 = x^4 y^4$

Rule (power of a fraction rule)

$$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$$

e.g. $\left(\frac{a}{b}\right)^4 = \frac{a^4}{b^4}$

Rule (zero exponent)

$$a^0 = 1$$

(Quotient rule $\frac{a^x}{a^y} = a^{x-y}$)

e.g. $\frac{a^4}{a^4}$ $a^{4-4} = a^0$

$$= 1$$

Rule (Negative Exponent)

$$a^{-x} = \frac{1}{a^{x}}$$

e.g. $a^{5} \times a^{-8} =$

$$a^{5} \times \frac{1}{a^{8}} = \frac{a^{5}}{a^{8}} \qquad a^{-3}$$

Rule (Fractional exponent)

$$a^{\frac{x}{y}} = \sqrt[y]{a^{x}}$$

e.g. $a^{5/2} = \sqrt{a^{5}}$

$$a^{5/4} = \sqrt[4]{a^{5}}$$

Expanding Brackets

example 1:

$$x^2(x+2)$$

$$x^2 \times x^1 = x^{2+1} = x^3$$

$$x^2 \times 2 = 2x^2$$

$$= x^3 + 2x^2$$

example 2:

$$(x+2)(x+3)$$

$$x^2 + 3x + 2x + 6$$

$$= x^2 + 5x + 6$$

example 3:

$$(x+y)(x-y)$$

$$x^2 \underline{- xy + xy} - y^2$$

$$= x^2 - y^2$$

$$= \underline{\underline{x^2 - y^2}}$$

example 4:

$$(3(x+2)+4)(x+1)$$

$$(3x+6+4)(x+1)$$

$$(3x+10)(x+1)$$

$$3x^2+3x+10x+10$$

$$=3x^2+13x+10$$

example 5:

$$(x+y)^2 = (x+y)(x+y)$$

$$=x^2+xy+xy+y^2$$

$$=x^2+2xy+y^2$$

Adding fractions

e.g. $\frac{x}{x+2} + \frac{1}{(x+2)^2}$.

Rule

denominators must be equal, then we add the numerators

$$\frac{x}{x+2} \times \frac{x+2}{x+2} = \frac{x(x+2)}{(x+2)^2}$$

$$\frac{x(x+2)+1}{(x+2)^2} = \frac{x^2+2x+1}{(x+2)^2}$$

e.g.

$$\frac{x(x+1)}{3} + \frac{2(x+2)}{1} + \frac{4}{1} =$$

$$\frac{x(x+1)}{3} + \frac{3(2(x+2))}{3} + \frac{4 \times 3}{3}$$

$$\frac{x(x+1)+6(x+2)+12}{3} = \frac{x^2+x+6x+24}{3}$$

Kahoot

$$= \frac{x^2+7x+24}{3}$$

Number Definitions

$\mathbb{Z}$ = Integer

- Natural number - the positive integers from 1 onwards.

Denoted as $\boxed{\mathbb{N}}$

e.g. 1, 2, 3, 5, 100 etc.

- Real number - includes whole numbers, rational numbers, irrational numbers, they can be +/- or 0.
  - They are real as they are NOT imaginary.

Denoted as $\boxed{\mathbb{R}}$

e.g. 1, 5.32, $\frac{3}{4}$, -0.5, π, $\sqrt{2}$

4! = factorial number → $1 \times 2 \times 3 \times 4 = 24$

$3! = 1 \times 2 \times 3$

$15! = 1 \times 2 \times 3 \times 4 \times \ldots \times 15$

- Rational number - a number that can be made by dividing two integers, a & b, where b is not equal to zero

Denoted as $\boxed{\mathbb{Q}}$

$$\boxed{\frac{a}{b}}$$

$b \neq 0$

e.g. $\frac{1}{2}$, 1 $\left(\frac{1}{1}\right)$, 0.75 $\left(\frac{3}{4}\right)$, $\frac{1}{3}$, $\frac{4}{3}$

- Irrational number - a real number that can **NOT** be made by dividing two integers, it's decimals goes on forever without repeating

Denoted as $\boxed{\mathbb{Q}'}$

e.g. π, $\sqrt{2}$, $\frac{\sqrt{5}}{7}$, $\sqrt{11}$

- Imaginary number - a number that when squared gives a negative result.
  - Symbol often is i or j.
  - When we square a real number we always get a positive or zero result. Therefore, we imagine squaring a number and getting a negative result.

$2 \times 2 = 4$

$(2)^2$

α

$-2 \times -2 = 4$

$(-2)^2$

- 0 is also an imaginary number
- We use imaginary numbers to allow us to find solutions to many equations that don't have real number solutions

$\boxed{i^2 = -1}$ $\boxed{i = \sqrt{-1}}$ = definition of i

$i^0 = 1$

$i^1 = i$

$i^2 = -1$

$i^3 = i^2 \times i^1 = -1 \times i = -i$

$i^4 = i^3 \times i^1 = -i \times i = -i^2 = 1$

e.g. simplify $(3i)^2$ (using power of product rule $(xy)^a = x^a y^a$)

$3^2 i^2$

$= -9$

e.g. simplify $(-9)^{1/2}$ (use fractional exponent rule) $a^{1/x} = \sqrt[x]{a}$

$\sqrt{-9}$

$\sqrt{9 \times -1} = \sqrt{9} \times \sqrt{-1}$

$= 3i$

- Complex number - a number that is a combination of a real number and an imaginary number

Denoted as $\boxed{\mathbb{C}}$

$a + bi$ ← $\sqrt{-1}$

Real part

Imaginary part

Examples

$1+i$, $-\underline{2}+\underline{\pi} i$, $\sqrt{2} + \frac{i}{2}$

Adding complex numbers

When adding 2 complex numbers we add each part seperately

$$(a + bi) + (c + di)$$

$$= (a+c) + (b+d)i$$

Example/ $(3+2i) + (1+7i)$

$$(3+1)+(2i+7i)$$

$$= 4 + 9i$$

Example/ $(3+5i) + (4-3i)$

$$(3+4) + (5i-3i)$$

$$= 7 + 2i$$

Multiplying complex numbers

$$(a + bi)(c + di)$$

$$= ac + adi + cbi + bdi^2$$

Example/ $(3+2i)(1+7i)$

$$3 + 21i + 2i + 14\underline{i}^2$$

$$14 \times -1 = -14$$

$$= 3 + 23i - 14$$

$$= \underline{-11} + \underline{23i}$$

Example/ $(x+5i)(1+2i)$

$$x + \underline{2xi + 5i} + 10i^2$$

$$-10$$

$$\underline{x - 10} + \underline{(2x+5)}\,\underline{i}$$

Example / $(x+(3i)^2)(x-(3i)^2)$

$$(3i)^2 = 3^2 i^2 = -9$$

$$(x-9)(x+9)$$

$$x^2 + \underline{9x - 9x} - 81$$

$$= x^2 - 81$$

Dividing Complex Numbers

- When dividing we want to multiply the numerator & denominator by the conjugate of the denominator

swaping the sign

$$a-b \downarrow a+b$$

Example/ $\dfrac{2+3i}{4-5i}$

$$\frac{2+3i}{4-5i} \times \frac{4+5i}{4+5i} \qquad i^2=-1$$

$$\frac{(2+3i)(4+5i)}{(4-5i)(4+5i)} = \frac{8+10i+12i+15i^2}{16-25i^2}$$

$$= \frac{8+22i-15}{16+25} = \frac{-7+22i}{41} = \underline{\frac{-7}{41}} + \underline{\frac{22}{41}}i$$

Example/ $\dfrac{5+2i}{1+i}$

$$\frac{(5+2i)(1-i)}{(1+i)(1-i)} = \frac{5-5i+2i-2i^2}{1-i^2}$$

$$= \frac{7-3i}{2} = \underline{\frac{7}{2}} - \underline{\frac{3}{2}}i$$

kahoot

Sums and Products

Sum denoted by $\boxed{\sum}$ sigma

Product denoted by $\boxed{\prod}$ Pi

Example/ $\sum_{i=1}^{3} i$ $\quad (1)+(2)+(3)$

$=6$

Example/ $\sum_{i=1}^{3} \sum_{j=0}^{2} ij^{2}$

~~Rule~~

$$\boxed{\sum\sum xy = \sum x \sum y}$$

$(i\times 0^2)+(i\times 1^2)+(i\times 2^2)$

$= i+4i = 5i$ $\qquad \sum_{i=1}^{3} 5i$

$5(1)+5(2)+5(3) = 30$

$5+10+15$

Example / $\prod_{i=1}^{3} i$

$(1)\times(2)\times(3)=6$

Example / $\prod_{i=2}^{4} 3i$

$3(2)\times 3(3)\times 3(4)$

$=6\times 9\times 12$

$=648$

Example / $\prod^{2} \sum^{3} i^{i}$ | Rule we

$$\prod_{j=1}^{2} \left(j^1 + j^2 + j^3\right)$$

$$\left(1^1+1^2+1^3\right) \times \left(\overset{2}{2^1}+\overset{4}{2^2}+\overset{8}{2^3}\right)$$

$$= 3 \times 14 = 42$$

Can swap sums (Σ) around but NOT sums with products

$\Sigma\Pi \neq \Pi\Sigma$

Proofs by Induction

Example

$\overbrace{\forall n \in \mathbb{N}}^{(n\,=\,\text{natural number})}$:

Claim = $\forall n \in \mathbb{N}$:

$$\sum_{i=0}^{n} i^2 = \frac{2n^3+3n^2+n}{6}$$

Step 1 : Induction start $n=1$

left

$$\sum_{i=0}^{1} i^2 = (0)^2 + (1)^2$$
$$= \underline{1}$$

right

$$\frac{2(1)^3+3(1)^2+1}{6} = \frac{6}{6} = \underline{1}$$

Step 2 : Induction assumption

$$\sum_{i=0}^{n} i^2 = \frac{2n^3+3n^2+n}{6}$$

Step 3: Induction step

When $i = n+1$

$$\sum_{i=1}^{n+1} i^2 = \overbrace{\sum_{i=0}^{n} i^2}^{\text{step2}} + (n+1)^2$$

right side

$$\frac{2n^3+3n^2+n}{6} + (n+1)^2 = \frac{2n^3+3n^2+n+6(n^2+2n+1)}{6}$$

$$= \frac{2n^3+3n^2+n+6n^2+12n+6}{6} = \frac{2n^3+9n^2+13n+6}{6}$$

left side sub $n = n+1$

$$\frac{2(n+1)^3+3(n+1)^2+(n+1)}{6} = \frac{2n^3+6n^2+6n+2+3n^2+6n+3+n+1}{6}$$

$$= \frac{2n^3+9n^2+13n+6}{6}$$

Both sides equal so by proof of induction the claim is true for all of n

Example 2

(when n = natural

number)

Claim: $\forall n \in \mathbb{N}$:

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

Step 1: Induction start $n=1$

left side

$$\sum_{k=1}^{1} k^3 = 1^3 = 1$$

right side

$$\frac{1^2(1+1)^2}{4} = \frac{4}{4} = 1$$

Step 2: Induction assumption

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

so must be true for all of n

Step 3 : Induction step

$n = n+1$

$$\sum_{k=1}^{n+1} k^3 = \sum_{k=1}^{n} k^3 + (n+1)^3$$

Right side

$$\frac{n^2(n+1)^2}{4} + (n+1)^3$$

$$= \frac{n^2(n+1)^2 + 4(n+1)^3}{4}$$

$$= \frac{n^4 + 2n^3 + n^2 + 4n^3 + 12n^2 + 12n + 4}{4}$$

$$= \frac{n^4 + 6n^3 + 13n^2 + 12n + 4}{4}$$

left side :

left side

$n = n+1$

$$\frac{(n+1)^2((n+1)+1)^2}{4} = \frac{(n+1)^2(n+2)^2}{4}$$

$$\frac{n^4+4n^3+4n^2+2n^3+8n^2+8n+n^2+4n+4}{4}$$

$$= \frac{n^4+6n^3+13n^2+12n+4}{4}$$

Both sides equal hence

$$\sum_{k=1}^{n+1} k^3 = \frac{(n+1)^2(n+1)^2}{4}$$

By proof by induction the claim is true